2012 American Control Conference
Fairmont Queen Elizabeth, Montréal, Canada
June 27-June 29, 2012

Saturated RISE Feedback Control for Euler-Lagrange Systems

N. Fischer, Z. Kan, W. E. Dixon

***Abstract*—This paper examines saturated control of a class of uncertain, nonlinear Euler-Lagrange systems which includes time-varying and nonlinearly parametrized functions with bounded disturbances using a continuous control law utilizing smooth saturation functions. The bounds on the control are known a priori and can be adjusted by changing the feedback gains. The saturated controller is shown to guarantee semi-global asymptotic tracking despite uncertainties in the dynamics.**

I. Introduction

Robust, high gain controllers [1]–[3] have been shown to be effective for systems with unstructured parametric uncertainties and bounded disturbances. Most high gain control techniques do not take into account the fact that the commanded input (a function of the system states) may require more force than is physically possible (e.g., due to large initial condition offsets, an aggressive desired trajectory, or some other disturbance). Because degraded control performance and the potential risk of thermal or mechanical failure can occur when unmodeled actuator constraints are violated, control schemes which ensure performance while actuator limits are not surpassed are motivated.

A significant amount of previous work on the design of controllers with bounded inputs has targeted the set-point (regulation) control problem [4]–[9]. This paper will focus on bounded input controllers for the more general tracking control problem. In [10], the authors propose an adaptive, full-state feedback controller to produce semi-global asymptotic tracking while compensating for unknown parametric uncertainties. The authors of [11] were able to extend the work of [6] to the tracking control problem by utilizing a general class of saturation functions to achieve a global uniform asymptotic tracking result for a linearly parametrizable system. Anti-windup schemes have been developed [12] to compensate for saturation nonlinearities in nonlinear Euler-Lagrange systems. To compensate for uncertain dynamics, Alvarez-Ramirez, et. al in [13] included an additional saturated integral term yielding a semi-global stability result. More recently in [14], a saturated PID framework controller was proposed which used sigmoidal functions to achieve a global asymptotic tracking result.

While each of the mentioned contributions propose saturated controllers with asymptotic stability results, the methods do not comment on their application to systems with both uncertain dynamics and additive unmodeled disturbances. General control literature suggests that robust techniques (such as high gain, sliding mode, or variable structure control) have successfully been developed to accommodate for parametric uncertainties and disturbances in the system [15], [16]. To address the inherent actuator saturation risks of robust controllers, Hong and Yao proposed the development of a continuous saturated adaptive robust control (SARC) algorithm [17] capable of achieving an ultimately bounded tracking result in the presence of an external disturbance. Corradini, et. al proposed a discontinuous saturated sliding mode controller [18] for linear plant models in the presence of bounded matched uncertainties to achieve a semi-global tracking result. In [19], two discontinuous control algorithms are proposed for robust stabilization of spacecraft in the presence of control input saturation, parametric uncertainty and external disturbances using a discontinuous variable structure control design. In [20], the authors develop a SARC controller a using discontinuous projection method to achieve globally bounded tracking of artificial muscles. However, while each of the previous results are able to address uncertain nonlinear systems with additive disturbances using a saturated control scheme, the use of discontinuous signals introduces limitations such as the demand for infinite bandwidth and potential chattering and motivates the design of other saturated robust control techniques. Robust control designs utilizing nested saturation functions for uncertain feedforward nonlinear systems [21]–[23] have guaranteed global asymptotic stability despite unmodeled dynamic disturbances; however, attempts at designing a transformation to convert an Euler-Lagrange system into a forward-complete system have required model knowledge [24]. This implies that methods developed for feedforward systems may not be applicable to uncertain Euler-Lagrange dynamics.

To overcome the limitations of sliding mode controllers, a control strategy called the robust integral of the sign of the error (RISE) was developed in [25] that contains a unique integral signum term resulting in a continuous solution which can accommodate for sufficiently smooth bounded disturbances and unstructured parametric uncertainty to achieve an asymptotic tracking result under the assumption that the disturbances are C^2 with bounded time derivatives. However, the RISE controller is also considered a high gain strategy and it too suffers from the risk of potentially requiring large control requirements which can lie outside the range of actuation limits.

This paper focuses on a new RISE strategy which consists

This research is supported by NSF award numbers 0547448, 0901491.
Nicholas Fischer, Zhen Kan, and Warren E. Dixon are with the Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, FL, USA. Email: {nic.fischer, kanzhen0322, wdixon}@ufl.edu.

of a saturated, continuous tracking controller for a class of uncertain, nonlinear systems which includes time-varying and nonlinearly parametrized functions and unmodeled dynamic effects. Unlike previous results that apply nonlinear combinations of saturated functions for stabilizing the closed loop system, the proposed controller is based on smooth hyperbolic functions that can easily be implemented in real-time applications. The bound on the control is known a priori and can be adjusted by adjusting the feedback gains. The saturated controller is shown to guarantee semi-global asymptotic tracking despite parametric uncertainties (with the exception of the inertia matrix) and additive disturbances, and without the use of acceleration measurements.

II. Dynamic System

Consider a class of nonlinear Euler-Lagrange systems of the form:

$$M(q)\ddot{q} + V_m(q,\dot{q})\dot{q} + G(q) + F(\dot{q}) + d(t) = u(t) \quad (1)$$

where $M(q) \in \mathbb{R}^{n\times n}$ denotes a generalized, state-dependent inertia matrix, $V_m(q,\dot{q}) \in \mathbb{R}^{n\times n}$ denotes a generalized centripetal-Coriolis matrix, $G(q) \in \mathbb{R}^n$ denotes a generalized gravity vector, $F(\dot{q}) \in \mathbb{R}^n$ denotes generalized friction, $d(t) \in \mathbb{R}^n$ denotes a general nonlinear disturbance (e.g., unmodeled effects), $q(t), \dot{q}(t), \ddot{q}(t) \in \mathbb{R}^n$ denote the generalized states and $u(t) \in \mathbb{R}^n$ denotes the generalized control force. The subsequent development is based on the assumptions that $q(t)$ and $\dot{q}(t)$ are measurable, and $V_m(q,\dot{q})$, $G(q)$, $F(\dot{q})$, $d(t)$ are unknown. In addition, the following assumptions will be exploited:

Assumption 1. The inertia matrix $M(q)$ is symmetric positive-definite, and satisfies the following inequality $\forall y(t) \in \mathbb{R}^n$:

$$\underline{m}\,\|y\|^2 \le y^T M y \le \bar{m}(q)\,\|y\|^2 \quad (2)$$

where $\underline{m} \in \mathbb{R}$ is a known positive constant, $\bar{m}(q) \in \mathbb{R}$ is a known positive function, and $\|\cdot\|$ denotes the standard Euclidean norm.

Assumption 2. The nonlinear disturbance term and its first two time derivatives (i.e., $d(t), \dot{d}(t), \ddot{d}(t)$) are bounded by known constants.

Assumption 3. The desired trajectory $q_d(t) \in \mathbb{R}^n$ is designed such that $q_d(t), \dot{q}_d(t), \ddot{q}_d(t), q_d^{(3)}(t), q_d^{(4)}(t) \in \mathcal{L}_\infty$.

Assumption 4. The inertia matrix and its inverse are assumed to be known.

Remark 1. To aid the subsequent control design and analysis, the vector $Tanh(\cdot) \in \mathbb{R}^n$ and the matrix $Cosh(\cdot) \in \mathbb{R}^{n\times n}$ are defined as follows

$$Tanh(\xi) \triangleq [tanh(\xi_1), ..., tanh(\xi_n)]^T \quad (3)$$

$$Cosh(\xi) \triangleq diag\{cosh(\xi_1), ..., cosh(\xi_n)\} \quad (4)$$

where $\xi = [\xi_1, ..., \xi_n]^T \in \mathbb{R}^n$. Based on the definition of (3), the following inequalities hold $\forall \xi \in \mathbb{R}^n$ [10]:

$$\|\xi\|^2 \ge \sum_{i=1}^{n} ln(cosh(\xi_i)) \ge \frac{1}{2}tanh^2(\|\xi\|)$$
$$\|\xi\| > \|Tanh(\xi)\|, \qquad \|Tanh(\xi)\|^2 \ge tanh^2(\|\xi\|)$$
$$\xi^T Tanh(\xi) \ge Tanh^T(\xi)\,Tanh(\xi). \quad (5)$$

III. Control Objective

The objective is to design an amplitude-limited, continuous controller which ensures the system state $q(t)$ tracks a desired time-varying trajectory $q_d(t)$ despite system uncertainty and additive bounded disturbances. To quantify the control objective, a tracking error denoted $e_1(q,t) \in \mathbb{R}^n$ is defined as

$$e_1 \triangleq q_d - q. \quad (6)$$

Two filtered tracking errors, denoted $e_2(e_1, \dot{e}_1, e_f, t)$, $r(e_2, \dot{e}_2, t) \in \mathbb{R}^n$, are defined as

$$e_2 \triangleq \dot{e}_1 + \alpha_1 Tanh(e_1) + Tanh(e_f), \quad (7)$$
$$r \triangleq \dot{e}_2 + \alpha_2 Tanh(e_2) + \alpha_3 e_2 \quad (8)$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ denote constant positive control gains, $Tanh(\cdot)$ was defined in (3), and $e_f(e_1, e_2) \in \mathbb{R}^n$ is an auxiliary signal whose dynamics are given by

$$\dot{e}_f \triangleq Cosh^2(e_f)\{-\gamma_1 e_2 + Tanh(e_1) - \gamma_2 Tanh(e_f)\} \quad (9)$$

where $e_f(0,0) = 0$, $\gamma_1, \gamma_2 \in \mathbb{R}$ are constant positive control gains and $Cosh(\cdot)$ was defined in (4). The auxiliary signal r is introduced to facilitate the stability analysis and is not used in the control design since the expression in (8) depends on the unmeasurable generalized state $\ddot{q}(t)$.

IV. Control Development

An open-loop tracking error can be obtained by premultiplying the filtered tracking error in (8) by $M(q)$, and substituting from (1), (6), (7), and (9) to yield

$$Mr = S + R - u(t) - M\gamma_1 e_2 \quad (10)$$

where the auxiliary functions $S(e_1, e_2, e_f, t) \in \mathbb{R}^n$ and $R(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$ are defined as

$$\begin{aligned} S \triangleq\ & M\ddot{q}_d + V_m\dot{q} + G + F - S_d \quad (11)\\ & + M\alpha_1 Cosh^{-2}(e_1)[e_2 - \alpha_1 Tanh(e_1) - Tanh(e_f)]\\ & - M\gamma_2 Tanh(e_f) + M\alpha_2 Tanh(e_2)\\ & + M\alpha_3 e_2 + MTanh(e_1), \end{aligned}$$

$$R \triangleq S_d + d \quad (12)$$

and a desired trajectory dependent auxiliary term, $S_d(q_d, \dot{q}_d, \ddot{q}_d, t) \in \mathbb{R}^n$, defined as

$$S_d \triangleq M_d\ddot{q}_d + V_{md}\dot{q}_d + G_d + F_d, \quad (13)$$

is added and subtracted. In (13), M_d, V_{md}, G_d, F_d denote $M(q_d) \in \mathbb{R}^{n\times n}$, $V_m(q_d, \dot{q}_d) \in \mathbb{R}^{n\times n}$, $G(q_d) \in \mathbb{R}^n$, $F(\dot{q}_d) \in \mathbb{R}^n$, respectively.

Injecting a saturation term is the obvious way to limit the control authority below an a priori limit; however, difficulty arises in the stability analysis and the development of the error signals when nonsmooth saturation functions are included in the control signal. Based on the form of (10) and through an iterative stability analysis, the continuous controller, $u(t)$, is designed as

$$u \triangleq \gamma_1 Tanh(v) \tag{14}$$

where $v(e_1, e_2) \in \mathbb{R}^n$ is the generalized Filippov solution to the following differential equation

$$\begin{aligned}\dot{v} = {} & Cosh^2(v)\left[M\alpha_2 Tanh(e_2) + M\alpha_3 e_2 + \beta sgn(e_2)\right. \\ & \left. -\alpha_1 Cosh^{-2}(e_1)\, e_2 - \dot{M}e_2 + \gamma_2 M e_2\right],\end{aligned} \tag{15}$$

$v(0) = 0$, where $\beta \in \mathbb{R}$ is a positive constant control gain and $sgn(\cdot)$ is defined $\forall \xi \in \mathbb{R}^m = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_m \end{bmatrix}^T$ as

$$sgn(\xi) \triangleq \begin{bmatrix} sgn(\xi_1) & sgn(\xi_2) & \dots & sgn(\xi_m) \end{bmatrix}^T.$$

Using Filippov's theory of differential inclusions [26]–[29], the existence of solutions can be established for $\dot{v} \in K[h_1](v)$, where $h_1(e_2, t) \in \mathbb{R}^n$ is defined as the right-hand side of $\dot{v}$ in (15) and $K[h_1] \triangleq \bigcap_{\delta>0} \bigcap_{\mu S_m=0} \overline{co} h_1(B(v,\delta) - S_m)$, where $\bigcap_{\mu S_m=0}$ denotes the intersection of all sets S_m of Lebesgue measure zero, $\overline{co}$ denotes convex closure, and $B(v,\delta) = \{\varsigma \in \mathbb{R}^n \mid \|v - \varsigma\| < \delta\}$ [30], [31]. The differential equation given in (15) is continuous except for the Lebesgue measure zero set of times $t \in [t_0, t_f]$ when $e_2(e_1, \dot{e}_1, t) = 0$. Hence, the set of time-instances for which $\dot{v}(e_2, t)$ is not defined is Lebesgue negligible. The absolutely continuous solution $v(e_2, t) = v(e_2(t_0), t_0) + \int_{t_0}^{t} \dot{v}dt$ does not depend on the value of $\dot{v}$ on a Lebesgue negligible set of time-instances [32].

The derivative of (14) is then given by $\dot{u}(e_1, e_2, t) \in \mathbb{R}^n$, defined as

$$\begin{aligned}\dot{u} = {} & M\gamma_1\alpha_2 Tanh(e_2) + M\gamma_1\alpha_3 e_2 + \gamma_1\beta sgn(e_2) \\ & -\gamma_1\alpha_1 M Cosh^{-2}(e_1)\, e_2 - \gamma_1 \dot{M} e_2 + \gamma_1\gamma_2 M e_2.\end{aligned} \tag{16}$$

The closed-loop tracking error system can be developed by taking the time derivative of (10), substituting (16), and by adding and subtracting $Tanh(e_2)$ and e_2 to yield

$$\begin{aligned}M\dot{r} = {} & -\frac{1}{2}\dot{M}r + \tilde{N} + N_d - M\gamma_1 r \\ & -\gamma_1\beta sgn(e_2) - Tanh(e_2) - e_2\end{aligned} \tag{17}$$

where $\tilde{N}(e_1, e_2, r, e_f) \in \mathbb{R}^n$ and $N_d\left(q_d, \dot{q}_d, \ddot{q}_d, q_d^{(3)}, t\right) \in \mathbb{R}^n$ are defined as

$$\tilde{N} \triangleq -\frac{1}{2}\dot{M}r + \dot{\tilde{S}} + Tanh(e_2) + e_2 \tag{18}$$

and

$$N_d \triangleq \dot{R}. \tag{19}$$

In (18), $\dot{\tilde{S}}(e_1, e_2, e_f, t) \in \mathbb{R}^n$ is defined as $\dot{\tilde{S}} \triangleq \dot{S} - \alpha_1\gamma_1 M Cosh^{-2}(e_1)\, e_2 + \gamma_1\gamma_2 M e_2$ where the last two terms are from (16) and cancel with inverse terms inside of $\dot{S}$ (which arise due to $Tanh(e_f)$ terms inside S) to yield $\dot{\tilde{S}}$ free of direct use of the gain parameter γ_1. Remaining γ_1 terms in $\dot{\tilde{S}}$ are encapsulated within $Tanh(\cdot)$ functions and thus can be upper bounded by 1. The structure of (17) is motivated by the desire to segregate terms that can be upper bounded by state-dependent terms and terms that can be upper bounded by constants. By applying the Mean Value Theorem, an upper bound can be developed for the expression in (18) [25]:

$$\left\|\tilde{N}\right\| \leq \rho(\|x\|)\,\|x\| \tag{20}$$

where the bounding function $\rho(\cdot) \in \mathbb{R}$ is a positive, globally invertible function, and $x(e_1, e_2, r, e_f) \in \mathbb{R}^{5n}$ is defined as

$$\begin{aligned}x \triangleq {} & [Tanh^T(e_1),\ Tanh^T(e_2),\ e_2^T, \\ & r^T,\ Tanh^T(e_f)]^T.\end{aligned} \tag{21}$$

From Assumptions 2 and 3, the following inequality can be developed based on the expression in (19):

$$\|N_d\| \leq \zeta_{N_{d1}}, \quad \left\|\dot{N}_d\right\| \leq \zeta_{N_{d2}} \tag{22}$$

where $\zeta_{N_{d1}}, \zeta_{N_{d2}} \in \mathbb{R}$, are known positive constants.

To facilitate the subsequent stability analysis, let γ_1 be selected as

$$\gamma_1 \triangleq \frac{\gamma_a + \gamma_b}{\underline{m}} \tag{23}$$

where $\gamma_a, \gamma_b \in \mathbb{R}$ are positive gain constants. Additionally, let the auxiliary constant $\lambda \in \mathbb{R}$ be defined by

$$\begin{aligned}\lambda \triangleq {} & min\{\alpha_1 - \frac{1}{2},\ 2\alpha_2 + \alpha_3, \\ & \alpha_3 - \frac{1}{2} - \frac{\gamma_1^2\zeta^2}{4},\ \gamma_2 - \frac{1}{\zeta^2},\ \gamma_a\}\end{aligned} \tag{24}$$

where $\zeta \in \mathbb{R}$ is a known positive adjustable gain parameter. Provided the control gains $\gamma_1, \gamma_2, \alpha_1, \alpha_2, \alpha_3$ are selected as

$$\alpha_1 > \frac{1}{2}, \quad \alpha_2 > 0, \quad \alpha_3 > \frac{1}{2} + \frac{\gamma_1^2\zeta^2}{4}, \quad \gamma_2 > \frac{1}{\zeta^2} \tag{25}$$

then λ in (24) is a positive adjustable constant. In addition to the sufficient gain conditions in (25), β and γ_1 should be selected as

$$\beta\gamma_1 > \zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_3} \tag{26}$$

to facilitate the subsequent stability analysis. It can be noted that the sufficient condition on β is related to the bound on the disturbance conditions given in Assumption 2. Let the vector $z(e_1, e_2, r, e_f) \in \mathbb{R}^{4n}$ be defined as

$$z \triangleq \left[e_1^T,\ e_2^T,\ r^T,\ Tanh^T(e_f)\right]^T \tag{27}$$

and the vector $y(e_1, e_2, r, e_f, P) \in \mathbb{R}^{4n+1}$ be defined as

$$y \triangleq \begin{bmatrix} z^T & \sqrt{P} \end{bmatrix}^T. \tag{28}$$

In (28), the auxiliary function $P(e_2,t) \in \mathbb{R}$ is defined as the generalized Filippov solution to the following differential equation

$$\dot{P} \triangleq -r^T\left(N_d - \beta\gamma_1 sgn\left(e_2\right)\right), \quad (29)$$
$$P\left(e_2\left(t_0\right), t_0\right) \triangleq \beta\gamma_1 \sum_{i=1}^{n} \left|e_{2i}\left(t_0\right)\right| - e_2\left(t_0\right)^T N_d\left(t_0\right)$$

where the subscript $i = 1, 2, ..., n$ denotes the ith element of the vector. Similar to the development in (15), existence of solutions for $P(e_2,t)$ can be established using Filippov's theory of differential inclusions for $\dot{P} \in K[h_2](P)$, where $h_2(e_2,r,t) \in \mathbb{R}$ is defined as $h_2 \triangleq -r^T\left(N_d - \beta\gamma_1 sgn\left(e_2\right)\right)$. Provided the sufficient condition for β in (25) is satisfied, $P(e_2,t) \geq 0$ (See the Appendix for details).

V. Stability Analysis

Theorem. *Given the dynamics in (1), the controller given by (14) and (15) ensures semi-global asymptotic tracking in the sense that*

$$\left\|e_1(t)\right\| \to 0 \ as \ t \to \infty \ \forall y \in \mathcal{D}$$

such that $\mathcal{D}$ is a domain defined as

$$\mathcal{D} \triangleq \left\{y \in \mathbb{R}^{4n+1} \mid tanh\left(\|y\|\right) \leq \rho^{-1}\left(2\sqrt{\lambda\gamma_b}\right)\right\}, \quad (30)$$

provided the sufficient conditions in (25) and (26) are satisfied.

Proof: Let $V_L(y,t) : \mathcal{D} \times [0,\infty) \to \mathbb{R}$ be a positive-definite function defined as

$$V_L \triangleq \sum_{i=1}^{n} ln\left(cosh\left(e_{1i}\right)\right) + \sum_{i=1}^{n} ln\left(cosh\left(e_{2i}\right)\right) + \frac{1}{2}e_2^T e_2 + \frac{1}{2}r^T M r + \frac{1}{2}Tanh^T\left(e_f\right)Tanh\left(e_f\right) + P \quad (31)$$

where e_{1i}, e_{2i} denote the i-th element of the vector $e_1(t)$ and $e_2(t)$, respectively, and which satisfies the following inequalities:

$$\phi_1(y) \leq V_L(t) \leq \phi_2(y). \quad (32)$$

Based on (5) and (31), the continuous positive definite functions $\phi_1(y), \phi_2(y) \in \mathbb{R}$ in (32) are defined as $\phi_1(y) \triangleq \frac{1}{2}min\left\{1, \underline{m}\right\}tanh^2\left(\|y\|\right)$ and $\phi_2(y) \triangleq max\left\{\frac{1}{2}\bar{m}(q), \frac{3}{2}\right\}\|y\|^2$, where $\underline{m}$ and $\bar{m}(q)$ were introduced in (2).

Under Filippov's framework, a generalized Lyapunov stability theory can be used to establish strong stability of the closed-loop system $\dot{y} = h_3(y,t)$, where $h_3(y,t) \in \mathbb{R}^{4n+1}$ denotes the right-hand side of the closed-loop error signals. The time derivative of (31) exists almost everywhere (a.e.), i.e., for almost all $t \in [t_0, t_f]$, and $\dot{V}(y,t) \overset{a.e.}{\in} \dot{\tilde{V}}(y,t)$ where

$$\dot{\tilde{V}}_L = \bigcap_{\xi \in \partial V_L(y)} \xi^T K\left[\dot{e}_1^T \ \dot{e}_2^T \ \dot{r}^T \ \dot{e}_f^T \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \ 1\right]^T,$$

and ∂V is the generalized gradient of $V(y,t)$ [33]. Since $V_L(y)$ is a Lipschitz continuous regular function,

$$\dot{\tilde{V}}_L \subset \nabla V^T K\left[\dot{e}_1^T \ \dot{e}_2^T \ \dot{r}^T \ \dot{e}_f^T \ \frac{1}{2}P^{-\frac{1}{2}}\dot{P} \ 1\right]^T \quad (33)$$

where

$$\nabla V \triangleq [Tanh^T\left(e_1\right), \left(Tanh^T\left(e_2\right) + e_2^T\right), r^T M, Tanh^T\left(e_f\right)Cosh^{-2}\left(e_f\right), 2P^{\frac{1}{2}}, \frac{1}{2}r^T\dot{M}r]^T.$$

Using the calculus for $K[\cdot]$ from [31], and substituting (6)-(9), (16), and (17) into (33), yields

$$\begin{aligned}\dot{\tilde{V}}_L \subset\ & r^T\left[-\frac{1}{2}\dot{M}r + \tilde{N} + N_d - M\gamma_1 r - Tanh\left(e_2\right) - e_2\right] \\ & - r^T\left[\gamma_1\beta K\left[sgn\left(e_2\right)\right]\right] \\ & + Tanh^T\left(e_1\right)\left[e_2 - \alpha_1 Tanh\left(e_1\right) - Tanh\left(e_f\right)\right] \\ & + Tanh^T\left(e_2\right)\left[r - \alpha_2 Tanh\left(e_2\right) - \alpha_3 e_2\right] \\ & + e_2^T\left[r - \alpha_2 Tanh\left(e_2\right) - \alpha_3 e_2\right] \\ & + Tanh^T\left(e_f\right)\left[-\gamma_1 e_2 + Tanh\left(e_1\right)\right] \\ & + Tanh^T\left(e_f\right)\left[-\gamma_2 Tanh\left(e_f\right)\right] + \dot{P} + \frac{1}{2}r^T\dot{M}r \quad (34)\end{aligned}$$

where $K[sgn(e_2)] = SGN(e_2)$ [31] such that $SGN(e_{2_i}) = 1$ if $e_{2_i} > 0$, $[-1,1]$ if $e_{2_i} = 0$, and -1 if $e_{2_i} < 0$. Substituting (29), canceling common terms and rearranging the resulting expression yields

$$\begin{aligned}\dot{\tilde{V}}_L \overset{a.e.}{=}\ & -\alpha_1 Tanh^T\left(e_1\right)Tanh\left(e_1\right) - M\gamma_1 r^T r \\ & - \alpha_2 Tanh^T\left(e_2\right)Tanh\left(e_2\right) - \alpha_3 e_2^T e_2 \\ & - \gamma_2 Tanh^T\left(e_f\right)Tanh\left(e_f\right) + r^T\tilde{N} \\ & + Tanh^T\left(e_1\right)e_2 - Tanh^T\left(e_2\right)\alpha_3 e_2 \\ & - \gamma_1 Tanh^T\left(e_f\right)e_2 - \alpha_2 e_2^T Tanh\left(e_2\right) \quad (35)\end{aligned}$$

where the set in (34) reduces to the scalar equality in (35) since the RHS is continuous a.e., i.e, the RHS is continuous except for the Lebesgue measure zero set of times when $e_2(e_1, \dot{e}_1, t) = 0$ [30], [32]. Utilizing Assumption 1 and the definition of (8), (20), and (22), the expression in (35) can be upper bounded as

$$\begin{aligned}\dot{\tilde{V}}_L \overset{a.e.}{\leq}\ & -\alpha_1\left\|Tanh\left(e_1\right)\right\|^2 - \left(2\alpha_2 + \alpha_3\right)\left\|Tanh\left(e_2\right)\right\|^2 \\ & - \alpha_3\left\|e_2\right\|^2 - \gamma_2\left\|Tanh\left(e_f\right)\right\|^2 - \underline{m}\gamma_1\|r\|^2 \\ & + \rho\|x\|\|r\| + \left\|Tanh\left(e_1\right)\right\|\left\|e_2\right\| \\ & + \gamma_1\left\|Tanh\left(e_f\right)\right\|\left\|e_2\right\|. \quad (36)\end{aligned}$$

Applying applying Young's Inequality, completing the squares on $r(t)$, and grouping terms, the expression in (36) can be upper bounded by

$$\begin{aligned}\dot{\tilde{V}}_L \overset{a.e.}{\leq} & -\left(\alpha_1 - \frac{1}{2}\right)\|Tanh(e_1)\|^2 \\ & -(2\alpha_2+\alpha_3)\|Tanh(e_2)\|^2 \\ & -\left(\alpha_3 - \frac{1}{2} - \frac{\gamma_1^2\zeta^2}{4}\right)\|e_2\|^2 \\ & -\left(\gamma_2 - \frac{1}{\zeta^2}\right)\|Tanh(e_f)\|^2 \\ & -\gamma_a\|r\|^2 + \frac{\rho^2(\|x\|)\|x\|^2}{4\gamma_b} \end{aligned} \quad (37)$$

where the gain condition in (23) was applied. Provided the sufficient conditions in (25) are satisfied the expression can be rewritten as

$$\dot{\tilde{V}}_L \overset{a.e.}{\leq} -\left(\lambda - \frac{\rho^2(\|x\|)}{4\gamma_b}\right)\|x\|^2 \quad (38)$$

where λ was defined in (24), and x was defined in (21).

The expression in (38) can be reduced by utilizing (21) and (27) to yield

$$\dot{\tilde{V}}_L \overset{a.e.}{\leq} -\phi_3(\|z\|) \leq -U(y) \quad (39)$$

where $\phi_3(\|z\|) \in \mathbb{R}$ is defined as $\phi_3 \triangleq \left(\lambda - \frac{\rho^2(\|x\|)}{4\gamma_b}\right) tanh^2(\|z\|)$ and $U(y) \triangleq c \cdot tanh^2(\|z\|)$ for some positive constant c, is a continuous, positive semi-definite function defined on $\mathcal{D}$ (defined in (30)).

The inequalities in (32) and (39) can be used to show that $V_L \in \mathcal{L}_\infty$ in $\mathcal{D}$, hence, $e_1, e_2, r, e_f \in \mathcal{L}_\infty$ in $\mathcal{D}$. From (3), $Tanh(e_1), Tanh(e_2), Tanh(e_f) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Thus, from (7) and (8), $\dot{e}_1, \dot{e}_2 \in \mathcal{L}_\infty$ in $\mathcal{D}$. From (14) and (5), $u(t) \in \mathcal{L}_\infty$ in $\mathcal{D}$. Because $e_1, \dot{e}_1 \in \mathcal{L}_\infty$, $q, \dot{q} \in \mathcal{L}_\infty$ in $\mathcal{D}$ from Assumption 3. From the above statements, (17) can be used to show that $\dot{r} \in \mathcal{L}_\infty$ in $\mathcal{D}$. The definition in (9) shows that $\dot{e}_f \in \mathcal{L}_\infty$in $\mathcal{D}$. Thus, $\dot{z} \in \mathcal{L}_\infty$ in $\mathcal{D}$ and it can be shown that z is uniformly continuous (UC) in $\mathcal{D}$. Since z is UC, $\|z\|$ is also UC which implies that $tanh(\|z\|)$ is UC. The definitions of $U(y)$ and $z(t)$ can be used to prove that $U(y)$ is UC in $\mathcal{D}$. Let $\mathcal{S} \subset \mathcal{D}$ denote a set defined as

$$\mathcal{S} \triangleq \left\{ y \in \mathcal{D} \mid \phi_2 < \frac{1}{2} min\{1, \underline{m}\}\left(\rho^{-1}\left(2\sqrt{\lambda\gamma_b}\right)\right)^2 \right\}. \quad (40)$$

The region of attraction in (40) can be made arbitrarily large to include any initial conditions by increasing the control gain γ_b. From (39), $tanh(\|z\|) \to 0 \;as\; t \to \infty \, \forall y(0) \in \mathcal{S}$. Based on the definition of $z(t)$ in (27), it can be shown that

$$\|e_1(t)\| \to 0 \;as\; t \to \infty \, \forall y(0) \in \mathcal{S}.$$

■

Remark 2. An important feature of the controller given by (14) is its applicability to the case where constraints exist on the available control. Note that the control law is bounded since an upper bound can be explicitly obtained as

$$\|u\| \leq \sqrt{n} \cdot \gamma_1 \quad (41)$$

where n denotes the degree of u.

VI. CONCLUSION

This paper provides a continuous saturated controller for a class of uncertain nonlinear Euler-Lagrange systems which includes time-varying and nonlinearly parametrized functions and additive bounded disturbances. The bound on the control is known a priori and can be adjusted by changing the feedback gains. The saturated controller is shown to guarantee semi-global asymptotic tracking despite uncertainties in the dynamics using smooth hyperbolic functions without the use of acceleration measurements. Future work will examine extensions of the proposed saturated control scheme to the output feedback problem and the inclusion of an uncertain inertia matrix.

REFERENCES

[1] Z. Qu, *Robust Control of Nonlinear Uncertain Systems*. Wiley Inc., New-York, 1998.

[2] V. I. Utkin, *Sliding Modes in Control and Optimization*. Springer-Verlag, 1992.

[3] J. Slotine and W. Li, *Applied Nonlinear Control*. Prentice Hall, 1991.

[4] R. Colbaugh, E. Barany, and K. Glass, "Global regulation of uncertain manipulators using bounded controls," in *Proc. IEEE Int. Conf. Robot. Autom.*, vol. 2, 1997, pp. 1148–1155.

[5] R. Kelly and V. Santibanez, "A class of global regulators with bounded control actions for robot manipulators," in *Proc. IEEE Conf. Decis. Control*, vol. 3, 1996, pp. 3382–3387.

[6] A. Zavala-Rio and V. Santibanez, "A natural saturating extension of the PD-with-desired-gravity-compensation control law for robot manipulators with bounded inputs," *IEEE Trans. Robot.*, vol. 23, no. 2, pp. 386–391, 2007.

[7] H. Yazarel, C. C. Cheah, and H. C. Liaw, "Adaptive SP-D control of a robotic manipulator in the presence of modeling error in a gravity regressor matrix: theory and experiment," *IEEE Trans. Robot. Autom.*, vol. 18, no. 3, pp. 373–379, 2002.

[8] Y. X. Su, D. Sun, L. Ren, and J. Mills, "Integration of saturated PI synchronous control and PD feedback for control of parallel manipulators," *IEEE Trans. Robot*, vol. 22(1), pp. 202–207, 2006.

[9] V. Santibanez, R. Kelly, and M. Llama, "A novel global asymptotic stable set-point fuzzy controller with bounded torques for robot manipulators," *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 3, pp. 362–372, 2005.

[10] W. E. Dixon, D. M. Dawson, F. Zhang, and E. Zergeroglu, "Global exponential tracking control of a mobile robot system via a pe condition," in *Proc. IEEE Conf. Decis. Control*, Phoenix, Arizona, December 1999, pp. 4822–4827.

[11] E. Aguinaga-Ruiz, A. Zavala-Rio, V. Santibanez, and F. Reyes, "Global trajectory tracking through static feedback for robot manipulators with bounded inputs," *IEEE Trans. Control Syst. Technol.*, vol. 17, no. 4, pp. 934–944, 2009.

[12] F. Morabito, A. Teel, and L. Zaccarian, "Nonlinear antiwindup applied to Euler-Lagrange systems," *Robotics and Automation, IEEE Transactions on*, vol. 20, no. 3, pp. 526–537, 2004.

[13] J. Alvarez-Ramirez, V. Santibanez, and R. Campa, "Stability of robot manipulators under saturated PID compensation," *IEEE Trans. Control Syst. Technol.*, vol. 16, no. 6, pp. 1333–1341, Nov. 2008.

[14] Y. Su, P. Muller, and C. Zheng, "Global asymptotic saturated PID control for robot manipulators," *IEEE Trans. Control Syst. Technol.*, vol. 18, no. 6, pp. 1280–1288, 2010.

[15] C. Abdallah, D. Dawson, P. Dorato, and M. Jamshidi, "Survey of robust control for rigid robots," *IEEE Control System Mag.*, vol. 11, no. 2, pp. 24–30, 1991.

[16] K. Zhou and J. C. Doyle, *Essentials of Robust Control*. Prentice Hall, 1997.

[17] Y. Hong and B. Yao, "A globally stable high-performance adaptive robust control algorithm with input saturation for precision motion control of linear motor drive systems," *IEEE/ASME Trans. Mechatron.*, vol. 12, no. 2, pp. 198–207, 2007.

[18] M. Corradini, A. Cristofaro, and G. Orlando, "Robust stabilization of multi input plants with saturating actuators," *IEEE Trans. Autom. Control*, vol. 55, no. 2, pp. 419–425, 2010.
[19] J. D. Boskovic, S.-M. Li, and R. K. Mehra, "Robust adaptive variable structure control of spacecraft under control input saturation," *J. Guid. Contr. Dynam.*, vol. 24(1), pp. 14–22, 2001.
[20] L. Zhang, J. Xie, and D. Lu, "Adaptive robust control of one-link joint actuated by pneumatic artificial muscles," in *Proc. Conf. BioInfo. Biomed. Eng.*, 2007, pp. 1185–1189.
[21] M. Arcak, A. Teel, and P. Kokotovic, "Robust nonlinear control of feedforward systems with unmodeled dynamics," *Automatica*, vol. 37, no. 2, pp. 265–272, 2001.
[22] L. Marconi and A. Isidori, "Robust global stabilization of a class of uncertain feedforward nonlinear systems," *Systems & Control Letters*, vol. 41, no. 4, pp. 281–290, 2000.
[23] G. Kaliora and A. Astolfi, "Nonlinear control of feedforward systems with bounded signals," *Automatic Control, IEEE Transactions on*, vol. 49, no. 11, pp. 1975–1990, 2004.
[24] F. Mazenc and S. Bowong, "Tracking trajectories of the cart-pendulum system," *Automatica*, vol. 39, no. 4, pp. 677–684, 2003.
[25] B. Xian, D. M. Dawson, M. S. de Queiroz, and J. Chen, "A continuous asymptotic tracking control strategy for uncertain nonlinear systems," *IEEE Trans. Autom. Control*, vol. 49, pp. 1206–1211, 2004.
[26] A. Filippov, "Differential equations with discontinuous right-hand side," *Am. Math. Soc. Transl.*, vol. 42 no. 2, pp. 199–231, 1964.
[27] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*. Kluwer Academic Publishers, 1988.
[28] G. V. Smirnov, *Introduction to the theory of differential inclusions*. American Mathematical Society, 2002.
[29] J. P. Aubin and H. Frankowska, *Set-valued analysis*. Birkhäuser, 2008.
[30] D. Shevitz and B. Paden, "Lyapunov stability theory of nonsmooth systems," *IEEE Trans. Autom. Control*, vol. 39 no. 9, pp. 1910–1914, 1994.
[31] B. Paden and S. Sastry, "A calculus for computing Filippov's differential inclusion with application to the variable structure control of robot manipulators," *IEEE Trans. Circuits Syst.*, vol. 34 no. 1, pp. 73–82, 1987.
[32] R. Leine and N. van de Wouw, "Non-smooth dynamical systems," in *Stability and Convergence of Mechanical Systems with Unilateral Constraints*, ser. Lecture Notes in Applied and Computational Mechanics. Springer Berlin / Heidelberg, 2008, vol. 36, pp. 59–77.
[33] F. H. Clarke, *Optimization and nonsmooth analysis*. SIAM, 1990.

VII. Appendix

Lemma 3. *Given the differential equation in (29), $P(e_2,t) \geq 0$ if β satisfies*

$$\beta\gamma_1 > \zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_3}. \tag{42}$$

Proof: By using (8), integrating by parts, and regrouping yields

$$\begin{aligned}
&\int_0^t r^T(\tau)\left(N_d(\tau) - \beta\gamma_1 sgn\left(e_2(\tau)\right)\right)\mathrm{d}\tau = \\
&\quad \int_{t_0}^t \alpha_2 Tanh^T\left(e_2(\tau)\right)\left[N_d(\tau) - \beta\gamma_1 sgn\left(e_2(\tau)\right)\right]d\tau \\
&\quad + \int_{t_0}^t \alpha_3 e_2^T(\tau)\left[N_d(\tau) - \beta\gamma_1 sgn\left(e_2(\tau)\right)\right]d\tau \\
&\quad - \int_{t_0}^t \alpha_3 e_2^T(\tau)\left[\frac{1}{\alpha_3}\frac{\partial N_d(\tau)}{\partial\tau}\right]d\tau + e_2^T N_d(t) \\
&\quad - e_2^T(t_0) N_d(t_0) - \beta\gamma_1\sum_{i=1}^n\left|e_{2i}(t)\right| + \beta\gamma_1\sum_{i=1}^n\left|e_{2i}(t_0)\right|.
\end{aligned} \tag{43}$$

From (9) and (22), the expression in (43) can be upper bounded by

$$\begin{aligned}
&\int_0^t r^T(\tau)\left(N_d(\tau) - \beta\gamma_1 sgn\left(e_2(\tau)\right)\right)\mathrm{d}\tau \leq \\
&\quad \int_{t_0}^t \alpha_2\left\|Tanh\left(e_2(\tau)\right)\right\|\left[\zeta_{N_{d1}} - \beta\gamma_1\right]d\tau \\
&\quad + \int_{t_0}^t \alpha_3\left\|e_2(\tau)\right\|\left[\zeta_{N_{d1}} + \frac{\zeta_{N_{d2}}}{\alpha_3} - \beta\gamma_1\right]d\tau \\
&\quad + \left\|e_2(t)\right\|\left[\zeta_{N_{d1}} - \beta\gamma_1\right] + \beta\gamma_1\sum_{i=1}^n\left|e_{2i}(t_0)\right| \\
&\quad - e_2^T(t_0) N_d(t_0).
\end{aligned} \tag{44}$$

Thus, from (44), if β satisfies (42), then

$$\begin{aligned}
&\int_0^t r^T(\tau)\left(N_d(\tau) - \beta\gamma_1 sgn\left(e_2(\tau)\right)\right)\mathrm{d}\tau \leq \\
&\beta\gamma_1\sum_{i=1}^n\left|e_{2i}(t_0)\right| - e_2^T(t_0) N_d(t_0) \leq P\left(e_2(t_0), t_0\right).
\end{aligned} \tag{45}$$

Integrating both sides of (29) yields

$$\begin{aligned}
P(e_2,t) = &\, P\left(e_2(t_0), t_0\right) \\
&- \int_0^t r^T(\tau)\left(N_d(\tau) - \beta\gamma_1 sgn\left(e_2(\tau)\right)\right)\mathrm{d}\tau,
\end{aligned}$$

which indicates $P(e_2,t) \geq 0$ from (45). ■